Indian Statistical Institute, Bangalore Centre
M.Math. (I Year) : 2013-2014
Semester II : Backpaper Examination
Measure Theoretic Probability

Jan. 2014 Time: 3 hours. Maximum Marks : 100

Note: Notation and terminology are understood to be as used in class. State clearly the results you are using in your answers.

1. (15 marks) Let f be a real valued function on $(\Omega, \mathcal{B})$. Show that f is Borel measurable if and only if the set $A_r \triangleq \{\omega \in \Omega : f(\omega) > r\}$ belongs to $\mathcal{B}$, for every rational number r.

2. (15 marks) Let $\{g_n\}$ be a sequence of nonnegative measurable functions on a measure space $(\Omega, \mathcal{B}, \mu)$. Show that
$$\int_\Omega \Big(\sum_{n=1}^{\infty} g_n\Big)(\omega) d\mu(\omega) = \sum_{n=1}^{\infty} \Big(\int_\Omega g_n(\omega) d\mu(\omega)\Big).$$

3. (15+5=20 marks) (i) Let λ, μ be totally finite measures on $(\Omega, \mathcal{B})$. Show that $\lambda \ll \mu$ if and only if for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that $E \in \mathcal{B},\ \mu(E) < \delta(\epsilon)$ imply $\lambda(E) < \epsilon$.

   (ii) Show that (i) above might fail if λ is an infinite measure. Take λ to be the counting measure on the natural numbers, and $\mu(\{k\}) = 1/2^k,\ k = 1, 2, \cdots$.

4. (15 + 10 = 25 marks) (i) Let $f_n, n \geq 1, f, g$ be Borel measurable functions on a $\sigma-$finite measure space $(\Omega, \mathcal{B}, \mu)$. Suppose that g is integrable, $|f_n(\cdot)| \leq g(\cdot)$ for all $n \geq 1$, and $\{f_n\}$ converges in $\mu-$measure to f. Show that
$$\int |f_n(\omega) - f(\omega)| d\mu(\omega) \to 0, \text{ as } n \to \infty.$$

   (ii) Let $\{X_n\}$ be a sequence of random variables converging to X in probability. Show that $\{X_n\}$ converges to X in distribution.

5. (25 marks) Let $X_1, X_2, \cdots$ be independent random variables each having a uniform distribution over $(0,1)$. Let $\Phi(\cdot)$ denote the distribution function of the standard normal distribution. Express

$$\lim_{n\to\infty} P\Big(\sum_{i=1}^{n} X_i \leq \frac{n}{2} + \frac{\sqrt{n}}{2\sqrt{3}}\Big)$$

in terms of $\Phi(\cdot)$.